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Equivalence of two graphical calculi

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Abstract. We consider the integrable systems which are connected with separation of variables in the Helmholtz operator on the real Riemannian spaces of constant curvature. An isomorphism is given for these systems with a quantum hyperbolic Gaudin magnet. Using this isomorphism, the complete classification of all separable coordinate systems on the manifolds considered is provided by means of the corresponding L-operators for the Gaudin magnet.

1. Introduction

Let us consider the problem of separation of variables in the Helmholtz equation $(\alpha, \beta = 1, \ldots, n)$

$$H\Psi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial y_\alpha}\left(\sqrt{g}\, g_{\alpha\beta} \frac{\partial \Psi}{\partial y_\beta}\right) = E\Psi \tag{1.1}$$

where the Hamiltonian H describes the free motion of a quantum particle in the Riemannian space with metric $g^{\alpha\beta}$ $(g = \det(g^{\alpha\beta}))$. We will only consider the real n-dimensional positive definite Riemannian spaces of constant curvature: n-sphere S_n, Euclidean n-space E_n, and upper sheet of the double-sheeted hyperboloid H_n having a constant curvature equal to 1, 0 and -1, respectively:

(i) S_n, the set of real vectors $x = (x_1, \ldots, x_{n+1})$ which satisfy $\Sigma_{\alpha=1}^{n+1} x_\alpha^2 = 1$ and have infinitesimal distance $dx^2 = \Sigma_{\alpha=1}^{n+1} dx_\alpha^2$;

(ii) E_n, the set of real vectors $x = (x_1, \ldots, x_n)$ with infinitesimal distance $dx^2 = \Sigma_{\alpha=1}^{n} dx_\alpha^2$;

(iii) H_n, the set of real vectors $x = (x_0, x_1, \ldots, x_n)$ which satisfy $x_0^2 - \Sigma_{\alpha=1}^{n} x_\alpha^2 = 1$, $x_0 > 1$, and have infinitesimal distance $dx^2 = dx_0^2 - \Sigma_{\alpha=1}^{n} dx_\alpha^2$.

Separation of variables (e.g. see [1]) indicates solution of equation (1.1) for the wavefunction Ψ in the following product form:

$$\Psi = \prod_{\alpha=1}^{n} \Psi_\alpha(z_\alpha; c_1, \ldots, c_n) \tag{1.2}$$

where Ψ_α depends only the variable z_α and on the complete set of quantum numbers c_α, which are the eigenvalues of mutually commuting integrals of motion. We will call orthogonal coordinate systems, in which equation (1.1) allows variable separation, s-systems (or s-coordinates or s-variables) from the abbreviation of the word *separable*. The main problem is to find all s-systems, to introduce corresponding separation variables, to carry out the procedure of variable separation, and to write explicitly the separation equations (i.e. ones for Ψ_α), which we then have to solve somehow.

The Helmholtz equation (1.1) with the g-metrics of an n-sphere, Euclidean n-space or n-hyperboloid has been met in a great number of works (see [1-5] and references therein). It is impossible to mention all those concerning separation of variables in this equation. We will refer here to [4] and [5], where the complete classification of all s-systems for the above manifolds is given.

Each s-system gives rise to a set of separation constants or integrals of motion. These integrals are quadratic functions on the generators of the Lie symmetry algebras of the spaces considered:

(i) $\mathrm{so}(n+1)$: $M_{\alpha\beta}=x_\alpha p_\beta - x_\beta p_\alpha \qquad \alpha,\beta=1,\ldots,n+1$

$$\mathrm{i}[M_{\alpha\beta}, M_{\gamma\delta}]=\delta_{\alpha\gamma}M_{\delta\beta}+\delta_{\alpha\delta}M_{\beta\gamma}+\delta_{\beta\gamma}M_{\alpha\delta}+\delta_{\beta\delta}M_{\gamma\alpha}$$

(ii) $\mathrm{e}(n)$: $M_{\alpha\beta}, p_\gamma \qquad \alpha,\beta,\gamma=1,\ldots,n$

$$\mathrm{i}[M_{\alpha\beta}, p_\gamma]=\delta_{\beta\gamma}p_\alpha-\delta_{\alpha\gamma}p_\beta \qquad [p_\alpha, p_\beta]=0$$

(iii) $\mathrm{so}(n,1)$: $M_{\alpha\beta}, M_{0\gamma}=x_0p_\gamma+x_\gamma p_0 \qquad \alpha,\beta,\gamma=1,\ldots,n$

$$\mathrm{i}[M_{\alpha\beta}, M_{0\gamma}]=\delta_{\beta\gamma}M_{0\alpha}-\delta_{\alpha\gamma}M_{0\beta} \qquad \mathrm{i}[M_{0\alpha}, M_{0\beta}]=M_{\alpha\beta}$$

where the variables x_α and p_β are canonical: $[p_\alpha, x_\beta]=-\mathrm{i}\delta_{\alpha\beta}$. The integrals of motion commute between themselves, and, due to full separation, their number is equal to the number of degrees of freedom. So we can say that each s-system is connected to the Liouville integrable system with a complete set of quadratic integrals of motion.

In [4, 5] the graphical technique to characterize all s-systems on S_n, E_n and H_n was developed. Each particular s-system was fixed by a graph. In the present paper we propose the equivalence of that graphical calculus to the one appearing in the well known problem of summing n quantum momenta. To recap, one must sum the n three-dimensional momenta s_α, $\alpha=1,\ldots,n$. If one knows how to sum the $k=2,\ldots,n$ momenta then one has a graph corresponding to the particular method of summing. To know how to sum is equivalent to knowing how to supplement the total momentum $J=\Sigma_{\alpha=1}^{n} s_\alpha$ by $n-2$ additional mutually commuting quadratic operators on s_α to obtain the complete set of operators.

In the present paper we give the algebraic interpretation of the classification in [4, 5]. We establish here the isomorphism between all the integrable systems, connected with all possible s-systems for the three spaces, and an n-site su(1, 1) Gaudin magnet. This Gaudin magnet is the integrable system, with a complete set of quadratic integrals of motion, given on the direct sum of the Lie algebras of rank 1: $\bigoplus_{\alpha=1}^{n}\mathrm{su}_\alpha(1,1)$. Using this isomorphism, the complete classification of separable coordinate systems is provided by means of the corresponding L-operators for the Gaudin magnet. The Gaudin magnet considered in this paper is the universal integrable system with quadratic integrals of motion having J as an extra integral of motion, so we can say that it has a direct relation with the summing of momenta.

2. The quantum hyperbolic Gaudin magnet

Let us consider the direct sum of Lie algebras of rank 1: $\mathscr{A}=\bigoplus_{\alpha=1}^{n}\mathrm{su}_\alpha(1,1)$. Generators s_α, $\alpha=1,\ldots,n$, of the $\mathscr{A}$-algebra satisfy the commutators

$$[s_\alpha^j, s_\beta^k]=-\mathrm{i}\delta_{\alpha\beta}\varepsilon_{jkl}g_{ll}s_\alpha^l \qquad g=\mathrm{diag}(1,-1,-1). \tag{2.1}$$

In the following we will use the g-metric to calculate the norm and scalar product of the operator vectors s_α:

$$s_\alpha^2 \equiv (s_\alpha, s_\alpha) = (s_\alpha^1)^2 - (s_\alpha^2)^2 - (s_\alpha^3)^2 \qquad (s_\alpha, s_\beta) = s_\alpha^1 s_\beta^1 - s_\alpha^2 s_\beta^2 - s_\alpha^3 s_\beta^3.$$

The Casimir operators of the $\mathscr{A}$-algebra (2.1) have the form $s_\alpha^2 = k_\alpha(k_\alpha - 1)$, where for the discrete series $k_\alpha = 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$, and for the representation of the universal enveloping group—the $\widetilde{\mathrm{SU}}(1,1)$ group—k_α varies continuously from zero to infinity: $0 < k_\alpha < \infty$.

We will define as the hyperbolic Gaudin magnet [6–9] the following quantum integrable Hamilton system on $\mathscr{A}$ given by n commuting integrals of motion H_α:

$$H_\alpha = 2 \sum_{\beta=1}^{n}{}' \frac{(s_\alpha, s_\beta)}{e_\alpha - e_\beta} \qquad [H_\alpha, H_\beta] = 0. \tag{2.2}$$

Here e_α are mutually non-coinciding real parameters of the model. To be more exact one has to call this model the n-site $\mathrm{su}(1,1) - XXX$ Gaudin magnet. Notice that all the H_α are quadratic functions on generators of the $\mathscr{A}$-algebra, and we have the equalities

$$\sum_{\alpha=1}^{n} H_\alpha = 0 \qquad \sum_{\alpha=1}^{n} e_\alpha H_\alpha = J^2 - \sum_{\alpha=1}^{n} s_\alpha^2$$

where the new variable

$$J = \sum_{\alpha=1}^{n} s_\alpha \tag{2.3}$$

is introduced, which is the total sum of the hyperbolic momenta s_α. The components of the vector J obey the su(1, 1) Lie algebra similarly to (2.1) and commute with all the H_α. The complete set of commuting integrals of motion is provided by the following choice: H_α, J^2, and, for example, $(J^3)^2$. The integrals (2.2) and (2.3) are generated by the 2×2 L-operator [7–9]

$$L(u) = \sum_{\alpha=1}^{n} \frac{1}{u - e_\alpha} \begin{pmatrix} \mathrm{i}s_\alpha^3 & -(s_\alpha^1 - s_\alpha^2) \\ -(s_\alpha^1 + s_\alpha^2) & -\mathrm{i}s_\alpha^3 \end{pmatrix} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \tag{2.4}$$

$$\begin{aligned} q\text{-}\det L(u) &= -A^2(u) - \tfrac{1}{2}\{B(u), C(u)\} \\ &= -\sum_{\alpha=1}^{n} \frac{H_\alpha}{u - e_\alpha} - \sum_{\alpha=1}^{n} \frac{s_\alpha^2}{(u - e_\alpha)^2} \end{aligned}$$

satisfying the standard linear algebra with the r-matrix being the operator P of permutation in $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$[\overset{1}{L}(u), \overset{2}{L}(v)] = \frac{1}{u - v}[P, \overset{1}{L}(u) + \overset{2}{L}(v)] \qquad P = \begin{pmatrix} 1000 \\ 0010 \\ 0100 \\ 0001 \end{pmatrix}. \tag{2.5}$$

Here we use the familiar notation for the tensor products of $L(u)$ and the 2×2 unit matrix I:

$$\overset{1}{L}(u) = L(u) \otimes I \qquad \overset{2}{L}(v) = I \otimes L(v).$$

It follows from equation (2.5) that q-det $L(u)$ is the generating function of the commuting integrals of motion:

$$[q\text{-det } L(u), q\text{-det } L(v)] = 0.$$

3. n-sphere S_{n-1}

Let us consider the following realization of the $\mathscr{A}$-algebra (2.1) in terms of the canonical operators $p_\alpha, x_\alpha([p_\alpha, x_\alpha] = -\mathrm{i}\delta_{\alpha\beta})$:

$$s_\alpha^1 = (p_\alpha^2 + x_\beta^2)/4 \qquad s_\alpha^2 = (p_\alpha^2 - x_\alpha^2)/4 \qquad s_\alpha^3 = \{p_\alpha, x_\alpha\}/4. \tag{3.1}$$

In this case the Casimir operators take the values $s_\alpha^2 = -\frac{3}{16}$, i.e. $k_\alpha = \frac{1}{4}, \frac{3}{4}$. Introduce the new operators

$$M_{\alpha\beta} = x_\alpha p_\beta - x_\beta p_\alpha \tag{3.2}$$

which are the generators of rotations in the space of vectors $x \in \mathbb{R}^n$. We have

$$(s_\alpha, s_\beta) = \tfrac{1}{8}(M_{\alpha\beta}^2 + \tfrac{1}{2}) \qquad \alpha \neq \beta. \tag{3.3}$$

The variables $M_{\alpha\beta}$ are the generators of the Lie algebra so(n):

$$\mathrm{i}[M_{\alpha\beta}, M_{\gamma\delta}] = \delta_{\alpha\gamma}M_{\delta\beta} + \delta_{\alpha\delta}M_{\beta\gamma} + \delta_{\beta\gamma}M_{\alpha\delta} + \delta_{\beta\delta}M_{\gamma\alpha}. \tag{3.4}$$

Equality (3.3) establishes a simple quadratic connection between the generators s_α of $\mathscr{A}$ and the $M_{\alpha\beta}$ of so(n). Under this isomorphism the integrals (2.2) transform into the following family of integrals, describing free motion on the n-sphere S_{n-1}:

$$H = \sum_{\alpha=1}^{n} h_\alpha H_\alpha = \frac{1}{4} \sum_{\alpha<\beta} \frac{h_\alpha - h_\beta}{e_\alpha - e_\beta} (M_{\alpha\beta}^2 + \tfrac{1}{2}). \tag{3.5}$$

For $h_\gamma = e_\gamma$ we get the Casimir operator of the so(n) algebra, $\Sigma_{\alpha<\beta} M_{\alpha\beta}^2$, and for $h_\gamma = e_\gamma^2$ we have the Hamiltonian of the quantum n-dimensional Euler–Manakov top,

$$H = \tfrac{1}{4} \sum_{\alpha<\beta} (e_\alpha + e_\beta)(M_{\alpha\beta}^2 + \tfrac{1}{2}) \tag{3.6}$$

possessing in our case the complete set of quadratic integrals of motion. Integrals (3.5) have the form of the Uhlenbeck integrals for the n-dimensional Neumann system in the case of a vanishing quadratic field, i.e. for the free motion on the n-sphere S_{n-1}.

Expressions for the components of total hyperbolic momentum J take the form

$$J^1 = \tfrac{1}{4}(p^2 + x^2) \qquad J^2 = \tfrac{1}{4}(p^2 - x^2) \qquad J^3 = \tfrac{1}{4} \sum_\alpha \{p_\alpha, x_\alpha\}. \tag{3.7}$$

Variables $M_{\alpha\beta}$ and J form the direct sum so$(n) \oplus$ su$(1,1)$,

$$[M_{\alpha\beta}, J^i] = 0 \qquad [J^j, J^k] = -\mathrm{i}\varepsilon_{jkl} g_{ll} J^l \qquad g = \mathrm{diag}(1, -1, -1).$$

Recall that the three variables J^i commute with all the H (equation (3.5)) and we can choose from them only two (additional to H) commuting integrals. Let them be the following quantities: the square of the total momentum,

$$J^2 = \tfrac{1}{4} \sum_{\alpha<\beta} M_{\alpha\beta}^2 + \tfrac{1}{16}(n^2 - 4n) \tag{3.8}$$

which is expressed through the Casimir operator of algebra (3.4), and, simultaneously, is equal to the Casimir operator of the su$(1,1)$ algebra (3.7), and the integral

$$2(J^1 - J^2) = x^2 = c \tag{3.9}$$

which gives us the equation of the n-sphere S_{n-1} $(c=1)$. Note that the presented quadratic isomorphism (3.1)-(3.9) between variables s_α and $M_{\alpha\beta}$ connects two integrable systems formulated on the different algebras (2.1) and (3.4). More exactly, the isomorphism is between the integrable system on the $\mathscr{A}$-algebra with generators s_α and the one on the so$(n)\oplus$su$(1, 1)$ algebra with generators $M_{\alpha\beta}$ and J.

Let us proceed to the separation of variables on the n-sphere S_{n-1} (3.9). Having the above-shown isomorphism of the two systems, we can easily carry out separation of variables using the L-operator (2.4) and the algebra (2.5) for the hyperbolic Gaudin magnet. The most general s-system on S_{n-1} is the system of ellipsoidal coordinates [4, 5], which is graphically pictured by the following 'irreducible' block:

e_1	e_2	....	e_n

(3.10)

where $e_\alpha \in \mathbb{R}$ are (as before) mutually non-coinciding real parameters, ordered in increase. The separation variables are defined as zeros of the off-diagonal element $B(u)$ of the L-matrix (2.4): $B(u_j)=0, j=1, \ldots, n-1$,

$$\sum_{\alpha=1}^{n} x_\alpha^2/(u-e_\alpha)=0 \Rightarrow u=u_j \qquad x_\alpha^2=c\frac{\Pi_{j=1}^{n-1}(u_j-e_\alpha)}{\Pi_{\beta\neq\alpha}(e_\beta-e_\alpha)}. \tag{3.11}$$

These ellipsoidal coordinates u_j on the sphere S_{n-1} satisfy the inequalities

$$e_1<u_1<e_2<u_2\ldots<u_{n-1}<e_n.$$

Each cell e_α of the block (3.10) gives rise to an item of the L-operator (2.4) in summing upon α. Notice here that $x_\alpha^2=2(s_\alpha^1-s_\alpha^2)$.

For each u_j let us define the additional variable v_j as follows:

$$v_j=-\mathrm{i}A(u_j)=\frac{1}{4}\sum_{\alpha=1}^{n}\frac{1}{u_j-e_\alpha}\{x_\alpha, p_\alpha\} \tag{3.12}$$

(the left substitution!). In what follows we will introduce the s-variables u_j and conjugate to them v_j as above: the first ones as zeros of $B(u)$, and the second ones as values of $A(u)$ in these zeros.

The changing of the variables s_α (or $M_{\alpha\beta}, J$) for the new variables v_j, u_j, c, and J^3 is the procedure of variable separation. One can rewrite the L-operator (2.4) with the new variables as

$$L(u)=\begin{pmatrix} A(u) & B(u)\\ C(u) & -A(u)\end{pmatrix} \qquad B(u)=-\frac{c}{2}\frac{\Pi_{j=1}^{n-1}(u-u_j)}{\Pi_{\alpha=1}^{n}(u-e_\alpha)} \tag{3.13}$$

$$A(u)=\frac{-2\mathrm{i}}{c}B(u)\left(J^3-\sum_{j=1}^{n-1}\frac{1}{u-u_j}D_jv_j\right)$$

where

$$D_j=-\frac{\Pi_{\gamma=1}^{n}(u_j-e_\gamma)}{\Pi_{i\neq j}(u_j-u_i)}=c\left(\sum_{\alpha=1}^{n}\frac{x_\alpha^2}{(u_j-e_\alpha)^2}\right)^{-1}>0. \tag{3.14}$$

The formula for $A(u)$ is obtained by interpolation with data (see equations (3.12), (2.4), (3.1) and (3.7),

$$A(u_j)=\mathrm{i}v_j \qquad j=1,\ldots,n-1 \qquad A(u)\xrightarrow[u\to\infty]{}\frac{\mathrm{i}}{u}J^3+\ldots$$

and the expression for $B(u)$ follows from the definition of the s-variables u_j (equation (3.11)).

Using the algebra (2.5) for the L-operator (2.4), one can prove the following commutators:

$$[c, J^3] = \mathrm{i}c \qquad [v_j, u_k] = -\mathrm{i}\delta_{jk}$$
$$[c, u_i] = [c, v_i] = [J^3, u_i] = [J^3, v_i] = 0 \qquad [u_i, u_k] = [v_i, v_k] = 0. \tag{3.15}$$

The procedure by which similar relations are derived from the algebra (2.5) is given in [8]. Having the interpolation (3.13) for $A(u)$ and $B(u)$, one can easily verify the following properties of conjugation of the introduced operators:

$$c^* = c \qquad (J^3)^* = J^3 \qquad u_i^* = u_i \qquad D_j v_j = v_j^* D_j. \tag{3.16}$$

From the v_j one can construct the self-conjugated operators w_j by the formulae

$$w_j = \sqrt{D_j}\, v_j \frac{1}{\sqrt{D_j}} \qquad v_j = \frac{1}{\sqrt{D_j}}\, w_j \sqrt{D_j}. \tag{3.17}$$

The operators c, J^3, u_i and w_i also obey the algebra (3.15). Equating the residues at $u = e_\alpha$ on the right- and left-hand sides of the interpolation for $A(u)$ (equation (3.13)) gives the equality

$$\tfrac{1}{4}\{p_\alpha, x_\alpha\} = \frac{x_\alpha^2}{c}\left(J^3 - \sum_{j=1}^{n-1} \frac{1}{e_\alpha - u_j} D_j v_j\right). \tag{3.18}$$

All the other explicit formulae for changing of variables can be listed, i.e. the connection between the two sets of $2n$ variables: p_α, x_α and u_j, v_j, c, J^3, where $\alpha = 1, \ldots, n$ and $j = 1, \ldots, n-1$.

A further problem is to find the separation equations. Let us substitute $u = u_j$ (the left substitution!) in the equation for q-det (equation (2.4)) to get the $n-1$ operator equalities:

$$-v_j^2 = \sum_{\alpha=1}^{n} \left(\frac{1}{u_j - e_\alpha} H_\alpha + \frac{1}{(u_j - e_\alpha)^2} s_\alpha^2\right). \tag{3.19}$$

We consider the spectral problem

$$H_\alpha \Psi = h_\alpha \Psi \tag{3.20}$$

where the Helmholtz operator (1.1) is the J^2 commuting with all the H_α. By action of the right- and left-hand sides of the operator equality (3.19) on Ψ, we obtain the $n-1$ equations of the form

$$v_j^2 \Psi + \sum_{\alpha=1}^{n} \left(\frac{h_\alpha}{u_j - e_\alpha} + \frac{\langle s_\alpha^2 \rangle}{(u_j - e_\alpha)^2}\right) \Psi = 0 \tag{3.21}$$

where the brackets $\langle\,\rangle$ denote the eigenvalue of an operator. In terms of w_j the above equations look like

$$w_j^2 \sqrt{D_j}\, \Psi + \sum_{\alpha=1}^{n} \left(\frac{h_\alpha}{u_j - e_\alpha} + \frac{\langle s_\alpha^2 \rangle}{(u_j - e_\alpha)^2}\right) \sqrt{D_j}\, \Psi = 0. \tag{3.22}$$

If we demand factorization of the wavefunction Ψ in accordance with condition (1.2) of variable separation in the form

$$\Psi = \sqrt{|V|} \prod_{i=1}^{n-1} \Psi_i(u_i) \tag{3.23}$$

where V is the Vandermond determinant,

$$V = \prod_{i<k} (u_i - u_k) \tag{3.24}$$

then the partial functions $\Psi_j(u_j)$ will satisfy the following separation equations:

$$-\frac{1}{\sqrt{P(u_j)}} \frac{\mathrm{d}^2}{\mathrm{d}u_j^2} (\sqrt{P(u_j)}\Psi_j(u_j)) + \sum_{\alpha=1}^{n} \left(\frac{h_\alpha}{u_j - e_\alpha} + \frac{\langle s_\alpha^2 \rangle}{(u_j - e_\alpha)^2} \right) \Psi_j(u_j) = 0 \tag{3.25}$$

where $P(u) = \Pi_{\alpha=1}^{n} (u - e_\alpha)$. Notice that the Vandermond determinant (3.24) appears to provide the invariant measure on the sphere $S_{n-1}(c = \boldsymbol{x}^2 = 1)$.

The equations obtained are separation equations for each of $n-1$ degrees of freedom which are connected by common eigenvalues of the integrals H_α. We did not carry out the rather complex procedure of standard separation of variables in the Helmholtz equation [1, 2, 4, 5], which gives the s-equations (3.25). Representation through 2×2 matrices (2.4) and (2.5), which is equivalent to the Lax pair, did allow us to simplify greatly the derivation of all the formulae and, in particular, the main equations (3.25) for partial functions.

In the following it will be more suitable to deal with the vector $\boldsymbol{L}(u)$ instead of the L-matrix (2.4):

$$\boldsymbol{L}(u) = \sum_{\alpha=1}^{n} \frac{\boldsymbol{s}_\alpha}{u - e_\alpha} \qquad L(u) = (\boldsymbol{\sigma}, \boldsymbol{L}(u)) \tag{3.26}$$

where

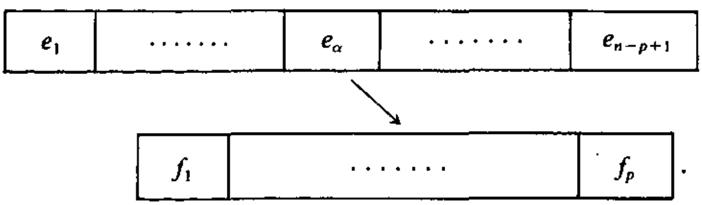
$$\sigma^1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}.$$

This vector satisfies the similar linear algebra (see equations (2.1) and (2.5))

$$[L^j(u), L^k(v)] = \frac{\mathrm{i}\varepsilon_{jkl} g_{lm}}{u - v} (L^m(u) - L^m(v)). \tag{3.27}$$

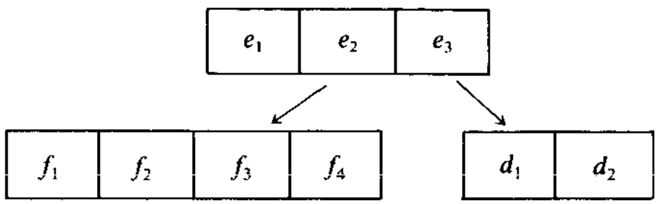
This algebra possesses the following important property (as well as the original algebra (2.5) of course): if we have $\boldsymbol{L}_1(u)$ and $\boldsymbol{L}_2(u)$, which both obey equation (3.27), and it is true that $[L_1^i(u), L_2^j(v)] = 0$, then $\boldsymbol{L}(u) = \boldsymbol{L}_1(u) + \boldsymbol{L}_2(u)$ also obeys equation (3.27).

Let us proceed now to the consideration of all possible degenerations of s-coordinates (3.10) [4, 5]. These coordinates were fixed by n different real parameters e_α. Degeneration means that some of them must coincide. The coinciding e_α can be pictured by a cell from which descends an arrow pointing at the 'internal structure' of this coincidence, which is the subblock with the parameters f_β, $\beta = 1, \ldots, p$:



The block with f_β can in turn consist of complex cells. In this case there must be attached to each such cell one arrow pointing at its 'internal structure', and so on. The general s-system on the n-sphere is fixed by a connected graph consisting of the blocks

(3.10) with the different parameters e_α, and these blocks are linked by arrows. In each block (apart from the highest) only one arrow can enter. From each block the arrows can descend: one from each complex cell. A cell is called *elementary* if no arrow descends from it. The dimension of the sphere S_{k-1} associated with the graph is calculated by the formula $k=$(number of elementary cells). The elementary vector s_α: $s_\alpha^2=-\frac{3}{16}$ is associated with each elementary cell, the complex vector s_α: $s_\alpha^2\neq-\frac{3}{16}$ is associated with each complex cell, and the elementary momenta, of which the complex vector consists, are indicated by arrows on the graph. Consider, for example, the graph

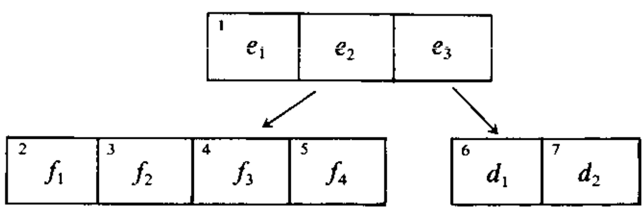


(3.28)

which gives some s-system on the sphere S_6. The momenta $s_{e_1}, s_{f_1}, s_{f_2}, s_{f_3}, s_{f_4}, s_{d_1}, s_{d_2}$ are elementary, but $s_{e_2}=\Sigma_{\alpha=1}^4 s_{f_\alpha}$, $s_{e_3}=\Sigma_{\beta=1}^2 s_{d_\beta}$ are complex. To each cell there corresponds a vector,

$$L_{e_1}(u)=s_{e_1}/(u-e_1) \qquad L_{e_2}(u)=s_{e_2}/(u-e_2),\ldots$$

and for all cells gathered in some block we have the vector $L(u)$ in the form of a sum (3.26). Each elementary vector is parametrized by the canonical variables x_α and p_α, in full analogy to the non-degenerate case (3.1). Let we deal with the sphere S_{n-1}. Then there exist n elementary momenta s_α, $\alpha=1,\ldots,n$, and formulae (3.1)-(3.4) are true. Introduce ordering of the elementary cells by writing down the integral number $\alpha=1,\ldots,n$ in increasing order for the corresponding elementary momentum as follows: first we go from left to right in the highest block, up to the first complex cell, then we descend to a subblock, where go again from left to right, up to the first complex cell in this subblock, and so on. If we arrive at the end of some subblock then we go up, and so on. For instance, for the graph (3.28) the ordering looks like



This graph corresponds to the following scheme for summation of the su(1, 1) momenta s_α, $\alpha=1,\ldots,7$:

$$J=[s_1+(s_2+s_3+s_4+s_5)+(s_6+s_7)] \tag{3.29}$$

where every pair of brackets fixes a block on the graph. With formulae of the same type as equation (3.29) the structure of graph can be restored uniquely and vice versa. The parameters of the graph together with its structure allow us to write the integrals of motion, the corresponding s-variables and the separation equations. For this we have to set the vector $L(u)$ to each block. The separation variables are determined as zeros of the off-diagonal element $B(u)$ of the matrix $L(u)$, or as zeros of $L^1(u)-L^2(u)$

in the case of the vector $L(u)$. They have different forms for different blocks. Let us list all the s-systems for the so(4) Lie algebra, i.e. on the sphere S_3:

(i) Jacobi elliptic coordinates,

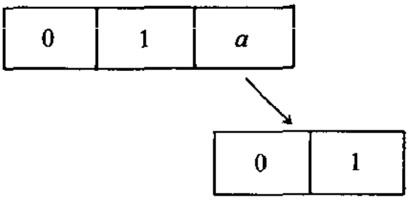
$$L(u)=s_1/u+s_2/(u-1)+s_3/(u-a)+s_4/(u-b)$$

$$H_1=M_{12}^2+M_{13}^2/a+M_{14}^2/b$$

$$H_2=M_{21}^2+M_{23}^2/(1-a)+M_{24}^2/(1-b).$$

(ii) Lamé rotational coordinates,

(*a*)

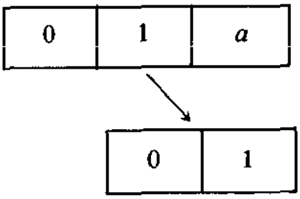


$$L_1(u)=s_1/u+s_2/(u-1)+(s_3+s_4)/(u-a)$$

$$L_2(u)=s_3/u+s_4/(u-1)$$

$$H_1=M_{34}^2 \qquad H_2=M_{13}^2+M_{14}^2+aM_{12}^2$$

(*b*)

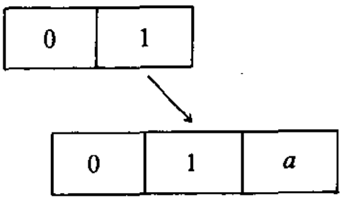


$$L_1(u)=s_1/u+(s_2+s_3)/(u-1)+s_4/(u-a)$$

$$L_2(u)=s_2/u+s_3/(u-1)$$

$$H_1=M_{23}^2 \qquad H_2=M_{14}^2+a(M_{12}^2+M_{13}^2).$$

(iii) Lamé subgroup reduction,

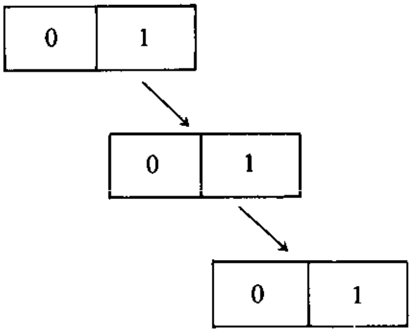


$$L_1(u)=s_1/u+(s_2+s_3+s_4)/(u-1)$$

$$L_2(u)=s_2/u+s_3/(u-1)+s_4/(u-a)$$

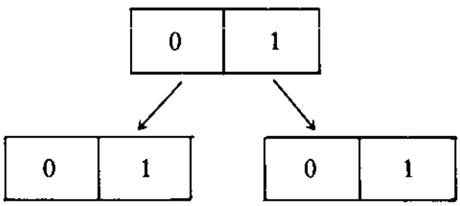
$$H_1=M_{12}^2+M_{13}^2+M_{14}^2 \qquad H_2=M_{24}^2+aM_{23}^2.$$

(iv) Spherical coordinates,



$$L_1(u) = s_1/u + (s_2 + s_3 + s_4)/(u-1)$$
$$L_2(u) = s_2/u + (s_3 + s_4)/(u-1)$$
$$L_3(u) = s_3/u + s_4/(u-1) \qquad H_1 = M_{23}^2 + M_{24}^2 \qquad H_2 = M_{34}^2.$$

(v) Cylindrical coordinates,



$$L_1(u) = (s_1 + s_2)/u + (s_3 + s_4)/(u-1)$$
$$L_2(u) = s_1/u + s_2/(u-1)$$
$$L_3(u) = s_3/u + s_4/(u-1) \qquad H_1 = M_{12}^2 \qquad H_2 = M_{34}^2.$$

All these six types of s-systems are exactly the six ways of (generalized) summing the four su(1, 1) momenta s_α:

(i) $J = (s_1 + s_2 + s_3 + s_4)$

(ii) (*a*) $J = (s_1 + s_2 + (s_3 + s_4))$

(*b*) $J = (s_1 + (s_2 + s_3) + s_4)$

(iii) $J = (s_1 + (s_2 + s_3 + s_4))$

(iv) $J = (s_1 + (s_2 + (s_3 + s_4)))$

(v) $J = ((s_1 + s_2) + (s_3 + s_4))$

where each pair of parentheses corresponds to a particular block on the graph. The integrals of motion are extracted from the $L_i^2(u)$ and the separation equations for the u_j look like

$$(v_j^2 + L_i^2(u_j))\Psi = 0 \tag{3.30}$$

where i is the index of a block, and j is the index of one of s-variables for this block.

Notice that in $L_i^2(u_j)$ in equation (3.30) the integrals H_α are replaced by their eigenvalues h_α. For the explicit form of these s-equations, see [2].

It is not difficult to derive also the formulae giving the explicit transformation to new variables. We recommend to the reader the original works [4, 5] for more details concerning graphical records of the s-systems. Notice that our interpretation of the graphs presented here seems to be more transparent than in the original works, because we associate the hyperbolic momentum with each cell of a graph, and these momenta exist because of the non-trivial (but simple in formulae) isomorphism of two integrable systems.

4. Euclidean space E_n

To proceed from the sphere S_{n-1} to the Euclidean space E_n one has to add the generators p_γ to the rotations $M_{\alpha\beta}$ (equation (3.2)), thus obtaining the e(n) Lie algebra for the generators $M_{\alpha\beta}, p_\gamma, \alpha, \beta, \gamma = 1, \ldots, n$. One has also to write three more formulae in addition to equations (3.1)-(3.4):

$$\begin{aligned} &s_\alpha^1 + s_\alpha^2 = p_\alpha^2/2 \qquad [p_\alpha, p_\beta] = 0 \\ &\mathrm{i}[M_{\alpha\beta}, p_\gamma] = \delta_{\beta\gamma} p_\alpha - \delta_{\alpha\gamma} p_\beta. \end{aligned} \tag{4.1}$$

Thus, a simple quadratic connection is established between the generators s_α of the $\mathscr{A}$-algebra and the generators of the $e(n)$ algebra. The complete classification of all s-systems on E_n [4, 5] includes two non-degenerate cases: ellipsoidal and paraboloidal coordinates, and, of course, their possible degenerations fixed by different graphs. Let us first consider the ellipsoidal ones. Introduce

$$L(u) = \sum_{\alpha=1}^{n} \frac{s_\alpha}{u - e_\alpha} - \frac{1}{4}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}. \tag{4.2}$$

The integrals of motion H_α in this case can be obtained as residues of $L^2(u)$. The ellipsoidal coordinates on E_n are defined as before, i.e. as zeros of $L^1(u) - L^2(u)$:

$$\sum_{\alpha=1}^{n} \frac{x_\alpha^2}{u - e_\alpha} = 1 \Rightarrow u = u_j \qquad x_\alpha^2 = \frac{\Pi_{j=1}^n (u_j - e_\alpha)}{\Pi_{\beta\neq\alpha} (e_\beta - e_\alpha)}. \tag{4.3}$$

The separation equations have the standard form (3.30). Such coordinates u_j are pictured by the following graph [4, 5]:

$$\left\langle \begin{array}{|c|c|c|c|} \hline e_1 & e_2 & \cdots\cdots & e_n \\ \hline \end{array} \right\rangle . \tag{4.4}$$

Let us proceed to the description of paraboloidal coordinates. Introduce

$$L(u) = \sum_{\alpha=2}^{n} \frac{s_\alpha}{u - e_{\alpha-1}} - \frac{1}{4}\begin{pmatrix} u - 2x_1 \\ -u + 2x_1 \\ -2p_1 \end{pmatrix}. \tag{4.5}$$

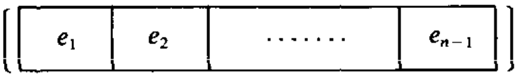
Paraboloidal coordinates on E_n are defined as zeros $L^1(u)-L^2(u)$:

$$\sum_{\alpha=2}^{n}\frac{x_\alpha^2}{u-e_{\alpha-1}}-u+2x_1=0\Rightarrow u=u_j \qquad j=1,\ldots,n$$

$$x_{\alpha+1}^2=-\frac{\Pi_{j=1}^{n}(u_j-e_\alpha)}{\Pi_{\beta\neq\alpha}(e_\beta-e_\alpha)} \qquad \alpha=1,\ldots,n-1 \tag{4.6}$$

$$x_1=\tfrac{1}{2}\left(\sum_{j=1}^{n}u_j-\sum_{\alpha=1}^{n-1}e_\alpha\right).$$

The cells e_α (s_α) collected in the paraboloidal block (4.5) we will picture as [4, 5]



(4.7)

The above-introduced n-dimensional orthogonal coordinates satisfy the following inequalities:

(i) ellipsoidal coordinates,

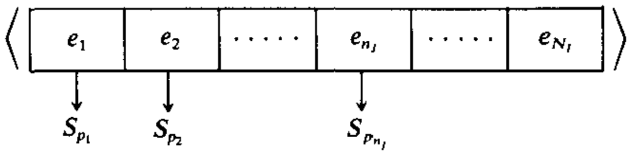
$$e_1<u_1<e_2<u_2\ldots<u_{n-1}<e_n<u_n$$

(ii) paraboloidal coordinates,

$$u_1<e_1<u_2<e_2\ldots<u_{n-1}<e_{n-1}<u_n.$$

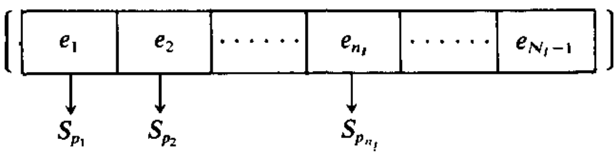
The general graph corresponding to an arbitrary s-system on E_n can be constructed as a sum of disconnected graphs of the type

(i)



or

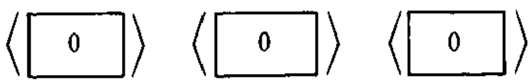
(ii)



where the complex momenta of blocks are as usual marked by arrows pointing out their 'internal structure', which is described by blocks of the sphere $S_{p_{n_j}}$.

As an example of this technique let us list all s-systems in the three-dimensional Euclidean space, i.e. on the e(3) Lie algebra. Notice that H_1 and H_2 given below are the additional integrals to p^2, which is the Casimir operator of e(3).

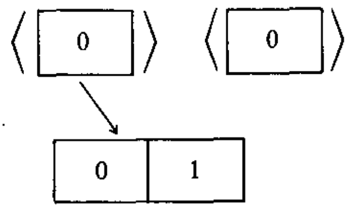
(i) Cartesian coordinates,



$$L_1(u)=s_1/u-\frac{1}{4}\begin{pmatrix}1\\-1\\0\end{pmatrix}\qquad L_2(u)=s_2/u-\frac{1}{4}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$L_3(u)=s_3/u-\frac{1}{4}\begin{pmatrix}1\\-1\\0\end{pmatrix}\qquad H_1=p_1^2\qquad H_2=p_2^2.$$

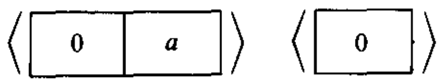
(ii) Cylindrical coordinates,



$$L_1(u)=(s_1+s_2)/u-\frac{1}{4}\begin{pmatrix}1\\-1\\0\end{pmatrix}\qquad L_2(u)=s_1/u+s_2/(u-1)$$

$$L_3(u)=s_3/u-\frac{1}{4}\begin{pmatrix}1\\-1\\0\end{pmatrix}\qquad H_1=M_{12}^2\qquad H_2=p_3^2.$$

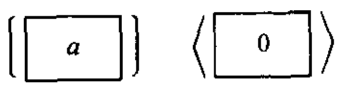
(iii) Elliptic cylindrical,



$$L_1(u)=s_1/u+s_2/(u-a)-\frac{1}{4}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$L_2(u)=s_3/u-\frac{1}{4}\begin{pmatrix}1\\-1\\0\end{pmatrix}\qquad H_1=M_{12}^2+ap_1^2\qquad H_2=p_3^2.$$

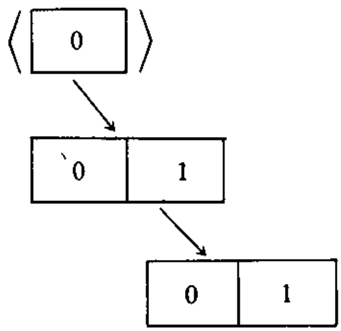
(iv) Parabolic cylindrical,



$$L_1(u)=s_2/(u-a)-\frac{1}{4}\begin{pmatrix}u-2x_1\\-u+2x_1\\-2p_1\end{pmatrix}\qquad L_2(u)=s_3/u-\frac{1}{4}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$H_1=\{M_{12},p_2\}-ap_2^2\qquad H_2=p_3^2.$$

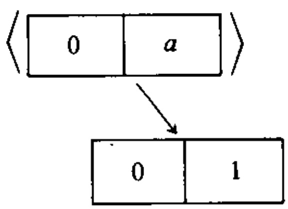
(v) Spherical,



$$L_1(u) = (s_1 + s_2 + s_3)/u - \frac{1}{4}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad L_2(u) = s_1/u + (s_2 + s_3)/(u-1)$$

$$L_3(u) = s_2/u + s_3/(u-1) \qquad H_1 = M_{12}^2 + M_{23}^2 + M_{31}^2 \qquad H_2 = M_{23}^2.$$

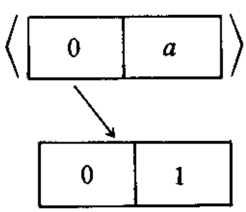
(vi) Prolate spheroidal,



$$L_1(u) = s_1/u + (s_2 + s_3)/(u-a) - \frac{1}{4}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad L_2(u) = s_2/u + s_3/(u-1)$$

$$H_1 = \tfrac{1}{2}(M_{12}^2 + M_{13}^2) + a p_1^2 \qquad H_2 = M_{23}^2.$$

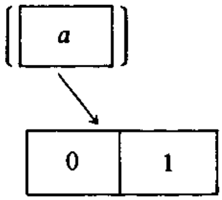
(vii) Oblate spheroidal,



$$L_1(u) = (s_1 + s_2)/u + s_3/(u-a) - \frac{1}{4}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad L_2(u) = s_1/u + s_2/(u-1)$$

$$H_1 = \tfrac{1}{2}(M_{13}^2 + M_{23}^2) - a p_3^2 \qquad H_2 = M_{12}^2.$$

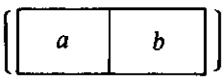
(viii) Parabolic,



$$L_1(u)=(s_2+s_3)/(u-a)-\frac{1}{4}\begin{pmatrix} u-2x_1 \\ -u+2x_1 \\ -2p_1 \end{pmatrix} \qquad L_2(u)=s_2/u+s_3/(u-1)$$

$$H_1=\{M_{12},p_2\}+\{M_{13},p_3\}+ap_1^2 \qquad H_2=M_{23}^2.$$

(ix) Parabolodial,

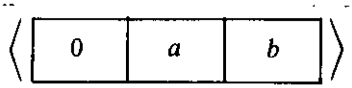


$$L(u)=s_2/(u-a)+s_3/(u-b)-\frac{1}{4}\begin{pmatrix} u-2x_1 \\ -u+2x_1 \\ -2p_1 \end{pmatrix}$$

$$H_1=\{M_{12},p_2\}+\{M_{13},p_3\}-ap_2^2-bp_3^2$$

$$H_2=M_{23}^2/(a-b)+\{M_{12},p_2\}-ap_2^2.$$

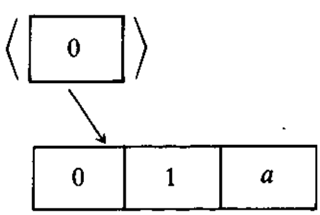
(x) Ellipsoidal,



$$L(u)=s_1/u+s_2/(u-a)+s_3/(u-b)-\frac{1}{4}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$H_1=M_{12}^2/a+M_{13}^2/b+p_1^2 \qquad H_2=M_{12}^2/a+M_{23}^2/(a-b)-p_2^2.$$

(xi) Conical,



$$L_1(u)=(s_1+s_2+s_3)/u-\frac{1}{4}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$L_2(u)=s_1/u+s_2/(u-1)+s_3/(u-a)$$

$$H_1=M_{12}^2+M_{23}^2+M_{31}^2 \qquad H_2=M_{13}^2/a+M_{23}^2/(a-1).$$

In full analogy to section 3, one can restore all the vectors $L_i(u)$ in the new variables u_j and v_j, thereby obtaining the explicit formulae of variables changing. Note that the classification of the s-systems presented for the Helmholtz equation in E_n,

$$p^2\Psi = E\Psi$$

using the classification of the corresponding L-operators, is simpler than the standard procedure of variable separation. The separation variables and equations are defined in the same way as in the previous section; in doing so we get the separation equations of different form for the different blocks in a graph. In contrast, for the non-degenerate block we obtain a unique separation equation for all s-variables, but considered on different intervals.

5. Hyperboloid H_n

To proceed from the sphere S_{n-1} to the hyperboloid H_n one has to add the hyperbolic rotations $M_{0\gamma} = x_0 p_\gamma + x_\gamma p_0$ to the $M_{\alpha\beta}$ (equation (3.2)), obtaining the so$(n, 1)$ Lie algebra for the generators $M_{\alpha\beta}$, $M_{0\gamma}$, $\alpha, \beta, \gamma = 1, \ldots, n$, which is the algebra of the isometry SO$(n, 1)$ group of H_n. One also has to write the following realization of the new hyperbolic momentum s_0 through the canonical operators x_0 and p_0 ($[p_0, x_0] = -i$):

$$\begin{aligned} &s_0^1 = -(p_0^2 + x_0^2)/4 \qquad s_0^2 = -(p_0^2 - x_0^2)/4 \qquad s_0^3 = \{p_0, x_0\}/4 \\ &(s_0, s_\alpha) = -(M_{0\alpha}^2 - \tfrac{1}{2})/8 \qquad (s_0, s_0) = -\tfrac{3}{16} \\ &\mathrm{i}[M_{\alpha\beta}, M_{0\gamma}] = \delta_{\beta\gamma} M_{0\alpha} - \delta_{\alpha\gamma} M_{0\beta} \qquad \mathrm{i}[M_{0\alpha}, M_{0\beta}] = M_{\alpha\beta}. \end{aligned} \tag{5.1}$$

Notice that, in contrast to the s_α, one must choose the discrete D^- series of the $\widetilde{\mathrm{SU}}(1, 1)$ representation for the s_0 (equation (5.1)). Thus, we have connected the generators s_α, s_0 to the generators $M_{\alpha\beta}$, $M_{0\gamma}$. The complete classification of all s-systems on H_n [5] is more complicated in comparison with the sphere and Euclidean space. There are four classes of s-systems here, $\mathfrak{U}$, $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$, and in the $\mathfrak{U}$-, $\mathfrak{C}$- and $\mathfrak{D}$-classes the number of different coordinate types increases with dimension. We describe all these systems with the help of the corresponding vectors $L(u)$ satisfying the algebra (3.27). The order of the description will follow the one fixed in the two previous sections. The algebraic interpretation presented allows us to formulate the theory for all s-systems on H_n in a compact and uniform way.

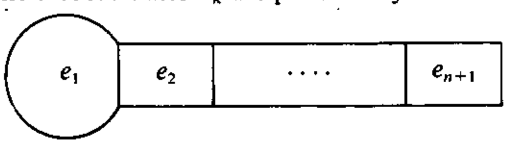
Let us begin with the class $\mathfrak{U}$. Introduce

$$L(u) = \sum_{\alpha=1}^{n} \frac{s_\alpha}{u - e_{\alpha+1}} + \frac{s_0}{u - e_1} \tag{5.2}$$

where the parameters $e_1, \ldots, e_{n+1}$ are in increasing order. The separation equations are written in the standard way:

$$(v_k^2 + L^2(u_k))\Psi = 0 \qquad k = 1, \ldots, n$$

where the variables v_k are defined as before as values of $L^3(u_k)$ (see equations (3.12) and (3.26)). The s-coordinates u_k are pictured by the following 'irreducible' block [5]:



(5.3)

Let us proceed to the description of the s-systems of class $\mathfrak{B}$. Introduce new variables:

$$x_\pm = \frac{1}{\sqrt{2}}(x_1 \pm \mathrm{i}x_0) \qquad p_\pm = \frac{1}{\sqrt{2}}(p_1 \pm \mathrm{i}p_0)$$
$$x_\pm^* = x_\mp \qquad p_\pm^* = p_\mp . \tag{5.4}$$

Notice that the $x_\pm$ and $p_\pm$ are canonical operators:

$$[p_+, p_-] = [x_+, x_-] = [x_\pm, p_\pm] = 0 \qquad [p_+, x_-] = [p_-, x_+] = -\mathrm{i}.$$

From them the non-self-adjoint hyperbolic momenta

$$s_{01} = \frac{1}{4}\begin{pmatrix} p_-^2 + x_+^2 \\ p_-^2 - x_+^2 \\ \{p_-, x_+\} \end{pmatrix} \qquad s_{01}^* = \frac{1}{4}\begin{pmatrix} p_+^2 + x_-^2 \\ p_+^2 - x_-^2 \\ \{p_+, x_-\} \end{pmatrix} \tag{5.5}$$

are constructed. The systems of class $\mathfrak{B}$ are fixed by following vector $\boldsymbol{L}(u)(a, b \in \mathbb{R})$:

$$\boldsymbol{L}(u) = \sum_{\alpha=2}^{n} \frac{\boldsymbol{s}_\alpha}{u - f_{\alpha-1}} + \frac{\boldsymbol{s}_{01}}{u - a - \mathrm{i}b} + \frac{\boldsymbol{s}_{01}^*}{u - a + \mathrm{i}b}. \tag{5.6}$$

The corresponding s-coordinates u_j are pictured by the following 'irreducible' block [5]:

$a + ib$	f_1	f_2	.....	f_{n-1}

. (5.7)

Consider now the s-systems of class $\mathfrak{C}$. Introduce the canonical operators $x_\pm$ and $p_\pm$ via

$$x_\pm = \frac{1}{\sqrt{2}}(x_1 \pm x_0) \qquad p_\pm = \frac{1}{\sqrt{2}}(p_1 \pm p_0)$$
$$[p_+, p_-] = [x_+, x_-] = [x_\mp, p_\pm] = 0 \qquad [p_+, x_+] = [p_-, x_-] = -\mathrm{i}. \tag{5.8}$$

The vector $\boldsymbol{L}(u)$ connected with s-systems of class $\mathfrak{C}$ has the form

$$\boldsymbol{L}(u) = \sum_{\alpha=2}^{n} \frac{\boldsymbol{s}_\alpha}{u - g_{\alpha-1}} + \frac{\boldsymbol{K}}{(u-a)^2} + \frac{\boldsymbol{M}}{u - a} \tag{5.9}$$

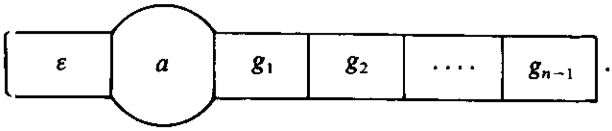
where parameters $g_1, \ldots, g_{n-1} \in \mathbb{R}$, in increasing order, do not coincide with each other and with the real parameter a. Vectors $\boldsymbol{K}$ and $\boldsymbol{M}$ are defined as follows:

$$\boldsymbol{K} = \frac{\varepsilon}{4}\begin{pmatrix} p_+^2 + x_-^2 \\ p_+^2 - x_-^2 \\ 2p_+x_- \end{pmatrix} \qquad \boldsymbol{M} = \boldsymbol{s}_0 + \boldsymbol{s}_1 \tag{5.10}$$

where $\varepsilon = \pm 1$. It is not difficult to verify that the $\boldsymbol{L}(u)$ (equation (5.9)) satisfies the algebra (3.27), and $\boldsymbol{K}$ plus $\boldsymbol{M}$ obey the following e(2, 1) algebra:

$$[K^i, K^j] = 0 \qquad [K^j, M^k] = -\mathrm{i}\varepsilon_{jkn} g_{nn} K^n$$
$$[M^j, M^k] = -\mathrm{i}\varepsilon_{jkn} g_{nn} M^n. \tag{5.11}$$

The coordinates u_j of class $\mathfrak{C}$ are pictured by the following 'irreducible' block [5]:



(5.12)

Consider now the final class $\mathfrak{D}$. We will use the $x_\pm$ and $p_\pm$ defined for class $\mathfrak{C}$ by equation (5.8). Let us introduce the vector $\boldsymbol{L}(u)$ for this class,

$$\boldsymbol{L}(u)=\sum_{\alpha=3}^{n}\frac{\boldsymbol{s}_\alpha}{u-h_{\alpha-2}}+\frac{\boldsymbol{I}}{(u-a)^3}+\frac{\boldsymbol{K}}{(u-a)^2}+\frac{\boldsymbol{M}}{u-a} \qquad (5.13)$$

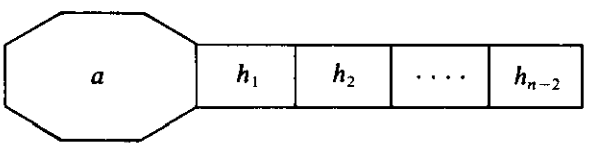
where the parameters $h_1,\ldots,h_{n-2}\in\mathbb{R}$, in increasing order, do not coincide with each other and with the real parameter a. Vectors $\boldsymbol{I}$, $\boldsymbol{K}$ and $\boldsymbol{M}$ have the form

$$\boldsymbol{I}=\frac{1}{2}\begin{pmatrix}p_+^2+x_-^2\\ p_+^2-x_-^2\\ 2p_+x_-\end{pmatrix} \qquad \boldsymbol{K}=\frac{1}{\sqrt{2}}\begin{pmatrix}p_+p_2+x_2x_-\\ p_+p_2-x_2x_-\\ p_+x_2+p_2x_-\end{pmatrix} \qquad \boldsymbol{M}=\boldsymbol{s}_0+\boldsymbol{s}_1+\boldsymbol{s}_2. \qquad (5.14)$$

It is easy to verify that $\boldsymbol{L}(u)$ (equation (5.13)) satisfies the algebra (3.27), and $\boldsymbol{I}$, $\boldsymbol{K}$ and $\boldsymbol{M}$ obey the following commutators:

$$\begin{aligned}&[I^j,I^n]=[I^j,K^n]=0 \qquad &&[K^j,K^n]=-\mathrm{i}\varepsilon_{jnp}g_{pp}I^p\\ &[I^j,M^n]=-\mathrm{i}\varepsilon_{jnp}g_{pp}I^p &&[K^j,M^n]=-\mathrm{i}\varepsilon_{jnp}g_{pp}K^p\\ &[M^j,M^n]=-\mathrm{i}\varepsilon_{jnp}g_{pp}M^p.\end{aligned} \qquad (5.15)$$

The s-coordinates u_j of the class $\mathfrak{D}$ are pictured diagrammatically as follows [5]:



(5.16)

In the cases $\mathfrak{U}$-$\mathfrak{D}$ there exists an additional integral, which is the total hyperbolic momentum:

$$\boldsymbol{J}=\boldsymbol{s}_0+\boldsymbol{s}_1+\ldots+\boldsymbol{s}_n. \qquad (5.17)$$

Its components take the form (cf equation (3.7))

$$\begin{aligned}&J^1=\tfrac{1}{4}\sum_{\beta=0}^{n}\varepsilon_\beta(p_\beta^2+x_\beta^2)\\ &J^2=\tfrac{1}{4}\sum_{\beta=0}^{n}\varepsilon_\beta(p_\beta^2-x_\beta^2) \qquad J^3=\tfrac{1}{4}\sum_{\beta=0}^{n}\{p_\beta,x_\beta\}\end{aligned} \qquad (5.18)$$

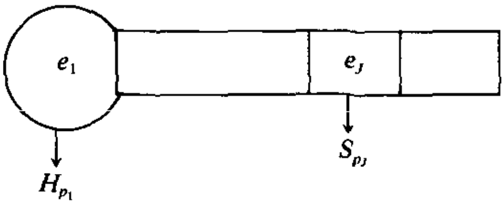
where $\varepsilon_\beta=-1$ if $\beta=0$, and $\varepsilon_\beta=1$ otherwise. Variables $M_{\alpha\beta}$, $M_{0\gamma}$, $\alpha,\beta,\gamma=1,\ldots,n$, and $\boldsymbol{J}$ constitute the direct sum of the Lie algebras so$(n,1)\oplus$su$(1,1)$. One can choose only two additional commuting integrals among three J^i. Let them be the square of the total momentum $\boldsymbol{J}^2$ and the integral

$$2(-J^1+J^2)=x_0^2-x_1^2-\ldots-x_n^2=c \qquad (5.19)$$

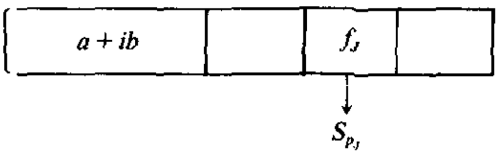
which gives the equation of a hyperboloid H_n ($c = 1$). Notice that two Casimir operators J^2 and $\Sigma_{\alpha<\beta}\, \varepsilon_\alpha M^2_{\alpha\beta}$ (of su(1, 1) and so(n, 1), respectively) are connected with each other, as for the sphere (equation (3.8)). Thus, we have established the isomorphism between integrable systems with all quadratic integrals of motion: that on the algebra $\mathscr{A}$ of the variables s_α (equation (2.1)) and that on the Lie algebra so(n, 1)$\oplus$su(1, 1) of the variables $M_{\alpha\beta}$, $M_{0\gamma}$ and J.

Let us now consider all possible degenerations of s-coordinates for the classes $\mathfrak{U}$–$\mathfrak{D}$ (equations (5.3), (5.7), (5.12) and (5.16)). The following schemes construct an arbitrary graph of s-systems on H_n with the help of the 'irreducible' blocks $\mathfrak{U}$–$\mathfrak{D}$ [5]:

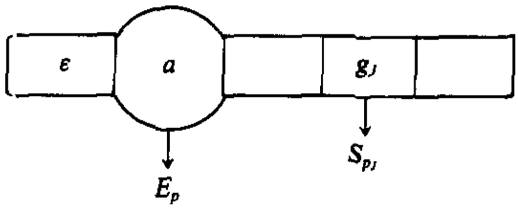
($\mathfrak{U}$)



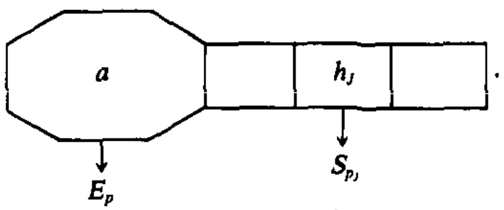
($\mathfrak{B}$)



($\mathfrak{C}$)



($\mathfrak{D}$)



The general graph of s-systems for H_n is constructed as the connected tree graph in accordance with the above rules. The symbol S_{p_J} indicates s-coordinates on the p_J-sphere. E_p indicates those on the Euclidean p-space and H_{p_1} denotes those on the p_1-hyperboloid. It is important to note that 'irreducible' blocks of the $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ types can enter in a given graph only once. The arrows descending from the different cells indicate degenerations, i.e. that the corresponding cell is complex; and these arrows indicate blocks revealing this degeneration. The arrows pointing to the block for the p_J-sphere were met in the two previous sections. New possibilities are in the $\mathfrak{U}$-, $\mathfrak{C}$- and $\mathfrak{D}$-classes, where arrows can point to subblocks of the hyperboloid H_{p_1} and of the Euclidean space E_p. In the first case one must join any possible graph for the hyperboloid H_{p_1} to the block of class $\mathfrak{U}$ in full analogy to the joining of a sphere. The second case demands special attention. Recall that the 'irreducible' blocks for E_n are of two types: ellipsoidal (equations (4.2) and (4.4)) or paraboloidal (equations

(4.5) and (4.7)). These blocks appear when degenerating the 'irreducible' blocks in the classes $\mathfrak{C}$ and $\mathfrak{D}$ in a slightly modified form. The corresponding modified L-operators look like

$$L(u)=\sum_{\alpha=i}^{k}\frac{\boldsymbol{s}_\alpha}{u-e_\alpha}-\boldsymbol{I} \tag{5.20}$$

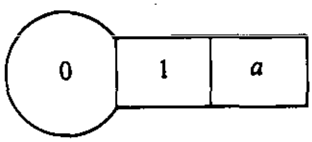
for the ellipsoidal block, and

$$L(u)=\sum_{\alpha=i+1}^{k}\frac{\boldsymbol{s}_\alpha}{u-e_\alpha}-\boldsymbol{I}u+\boldsymbol{K}_i \tag{5.21}$$

for the paraboloidal block, where $i=2$ for class $\mathfrak{C}$ and $i=3$ for class $\mathfrak{D}$. Vectors $\boldsymbol{I}$ and $\boldsymbol{K}_2\equiv\boldsymbol{K}$ are defined by formulae (5.14), and vector $\boldsymbol{K}_3$ coincides with $\boldsymbol{K}$ when changing x_2, p_2 on x_3, p_3. The arrow pointing to E_p will then indicate joining of the vector (5.20) or (5.21) plus their degenerations, analogously to the previous section. In the cells, which this arrow descends from, the vector $\boldsymbol{M}$ will be complex: $\boldsymbol{M}=\boldsymbol{s}_0+\boldsymbol{s}_1+\ldots+\boldsymbol{s}_k$.

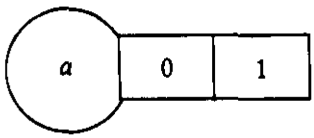
Let us illustrate this technique for the example of the two-dimensional hyperboloid H_2. The higher dimensions can be analysed in the same way. Notice that the H below are the additional integrals to the su(1, 1) Casimir operator $C=M_{01}^2+M_{02}^2-M_{12}^2$ on H_2.

(i) Elliptic coordinates,



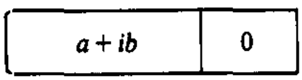
$$L(u)=\boldsymbol{s}_0/u+\boldsymbol{s}_1/(u-1)+\boldsymbol{s}_2/(u-a) \qquad H=M_{02}^2+aM_{01}^2.$$

(ii) Hyperbolic coordinates,



$$L(u)=\boldsymbol{s}_0/(u-a)+\boldsymbol{s}_1/u+\boldsymbol{s}_2/(u-1) \qquad H=M_{01}^2-aM_{12}^2.$$

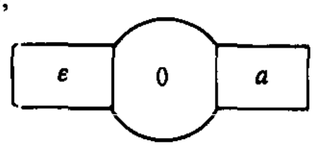
(iii) Semihyperbolic coordinates,



$$L(u)=\boldsymbol{s}_2/u+[\boldsymbol{s}_{01}/(u-a-ib)+\mathrm{cc}]$$

where $\boldsymbol{s}_{01}$ is given in equation (5.5), and $H=aM_{01}^2+b\{M_{12},M_{02}\}$.

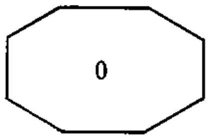
(iv)–(v) Hyperbolic parabolic coordinates ($\varepsilon=+1$), and elliptic parabolic coordinates ($\varepsilon=-1$),



$$L(u)=\frac{\boldsymbol{s}_2}{u-a}+\frac{\boldsymbol{K}}{u^2}+\frac{\boldsymbol{M}}{u} \qquad H=M_{01}^2+\frac{\varepsilon}{2a}(M_{12}-M_{02})^2$$

where vectors $\boldsymbol{K}$ and $\boldsymbol{M}$ are given in equation (5.10).

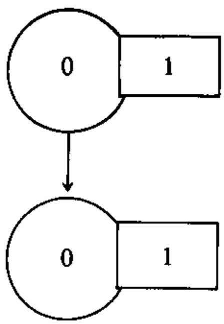
(vi) Semicircular parabolic coordinates,



$$L(u)=\frac{I}{u^3}+\frac{K}{u^2}+\frac{M}{u}$$

where the I, K, M are given in equation (5.14), and $H=\{M_{01}, M_{12}-M_{02}\}$.

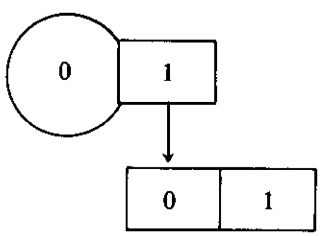
(vii) Equidistant coordinates,



$$L_1(u)=(s_0+s_1)/u+s_2/(u-1)$$

$$L_2(u)=s_0/u+s_1/(u-1) \qquad H=M_{01}^2.$$

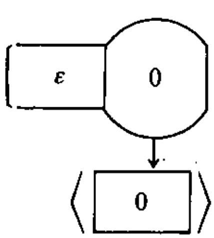
(viii) Spherical coordinates,



$$L_1(u)=s_0/u+(s_1+s_2)/(u-1)$$

$$L_2(u)=s_1/u+s_2/(u-1) \qquad H=M_{12}^2.$$

(ix) Horicyclic coordinates,



$$L_1(u)=\frac{\varepsilon}{2}\frac{I}{u^2}+\frac{M}{u} \qquad L_2(u)=\frac{s_2}{u}-I$$

where I and M are given in equation (5.14), and $H=(M_{12}-M_{02})^2$.

6. Discussion

In the present work the isomorphism between two large classes of integrable systems with all quadratic integrals of motion has been studied. The first family is the integrable systems connected with the separation of variables in the Helmholtz operator on the real Riemannian spaces of constant curvature, the second one is the simplest hyperbolic Gaudin magnet with various 'boundary terms'. For the cases of an n-sphere and an n-hyperboloid we have established the important fact that the trees (graphs) of s-systems on these spaces are equivalent to those appearing in the summation of n-hyperbolic momenta. This equivalence of different diagrammatic calculi gives us the following result: the unitary matrices between one 'tree' basis and another are simply the $3nj$ symbols of the su(1, 1) algebra. The majority of the literature has been devoted to calculations of such matrices (t-coefficients) but as far as I know this simple fact has not been observed before.

One of the important features in my opinion is the use of the linear r-matrix algebra and of the corresponding 2×2 L-operators. This formulation of quadratics on considered spaces is also new.

It should be noticed that the above s-coordinates are general orthogonal coordinate systems on three spaces and include as special case [4, 5] the 'tree' (poly- and horispherical) graphs due to Vilenkin [3].

As for further study, the author intends to consider complex spaces of constant curvature and other homogeneous symmetric spaces of rank 1. The above-described isomorphism was announced in [10] for the real three-dimensional sphere and in [11] for all manifolds in a classical setting (the Hamilton–Jacobi equation).

Acknowledgments

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